Introduction to Fractal Geometry

– The Wonder of the Noninterger Dimension –

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Complex Objects

Complex Objects

Clouds in the sky, thunderstorms, snow crystals,
trees, branches, leaves
coastlines, rivers, ridge lines, stones,
blood vessels, lungs, brains

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↓

fine structures at arbitrary small scales

Clouds

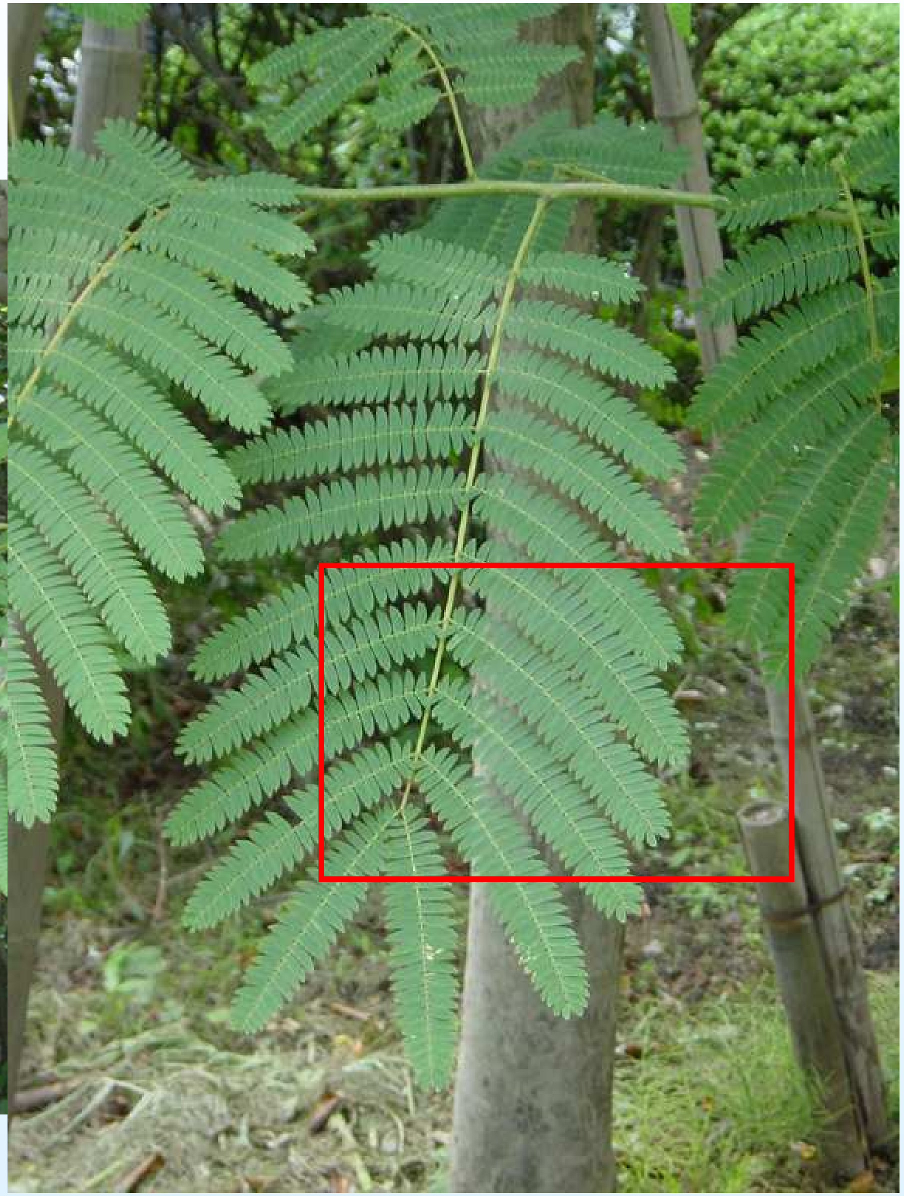
Clouds again

Leaves

Leaves

Leaves


Leaves

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fine structures at arbitrary small scales

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fractal (self-similar) structure

Fractal

Fractal

- complex geometric shapes with fine structure at arbitrary scales
- some degree of self-similarity
- Origin of the word
  - Fractal – Benoit Mandelbrot, 1975
  - frangere $\longrightarrow$ fractus of Latin
- Different from the Newtonian philosophy

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How to quantify fractal structures

$\Downarrow$

What is a good measure ?

Examples of self-similar fractals

- Cantor set
- von Koch curve
- Sierpiński gasket

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POINTS

1. Algorithms for producing these sets
2. Measures for quantifying the complexities of these sets

Length? Area? Volume? $\cdots$ or others...?

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No!

Fractal Dimension is a good measure.

Cantor set

Cantor set

- the closed interval $S_0 = [0, 1]$

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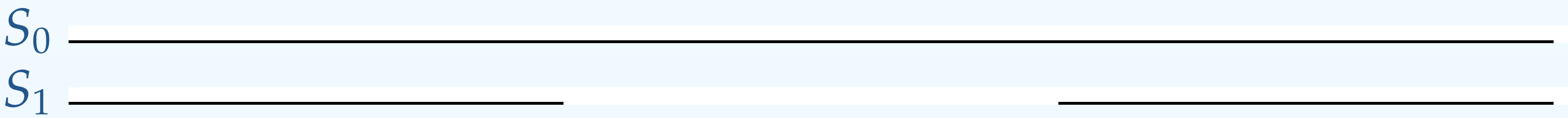
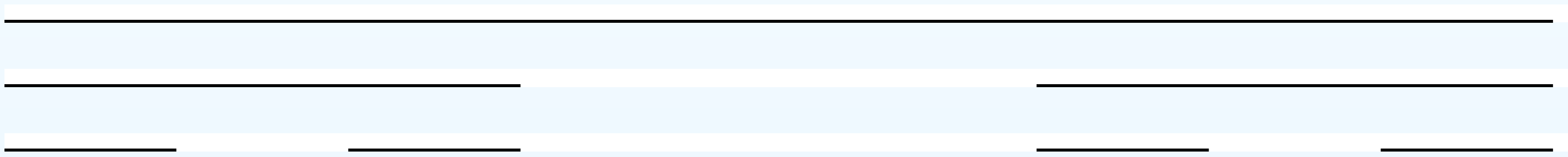
- remove its open middle third $[1/3, 2/3] \rightarrow S_1$



Cantor set

- the closed interval $S_0 = [0, 1]$
- remove its open middle third $[1/3, 2/3] \rightarrow S_1$
- remove the open middle thirds from the intervals $\rightarrow S_2$

S_0
S_1
S_2



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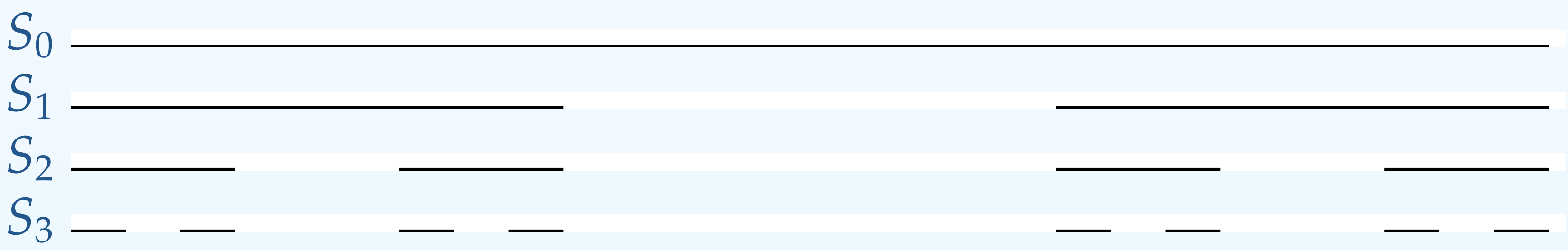

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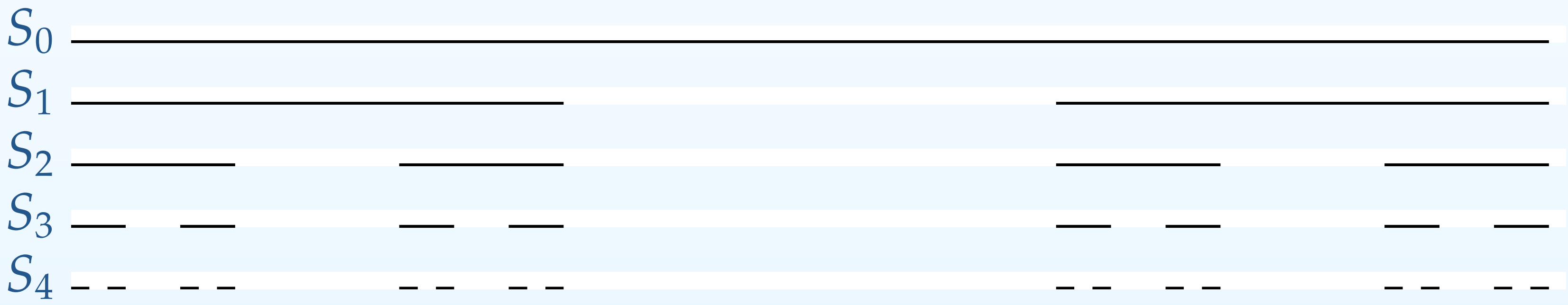

Cantor set

- the closed interval $S_0 = [0, 1]$
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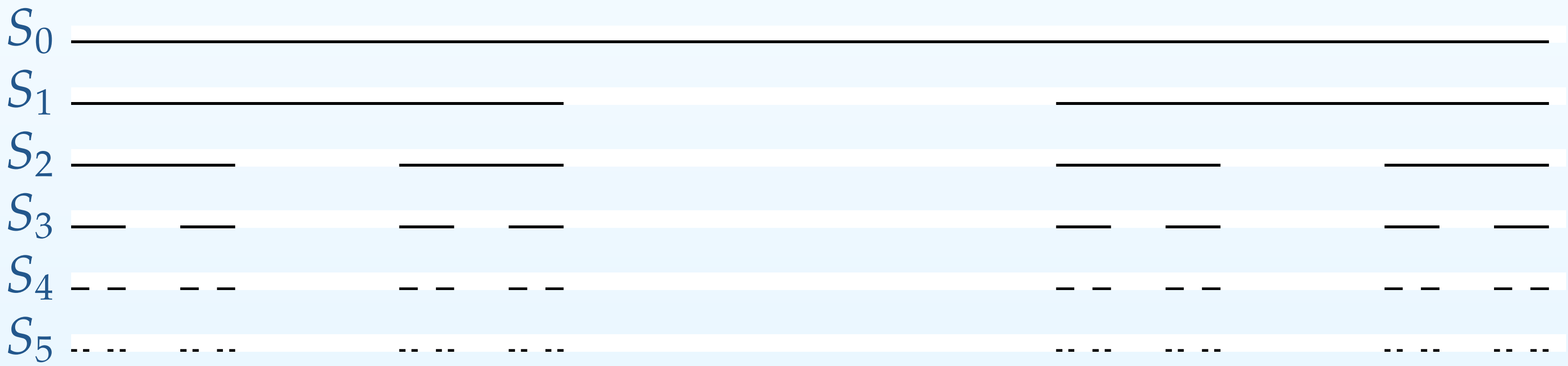
- remove the open middle thirds from the intervals $\to S_2$



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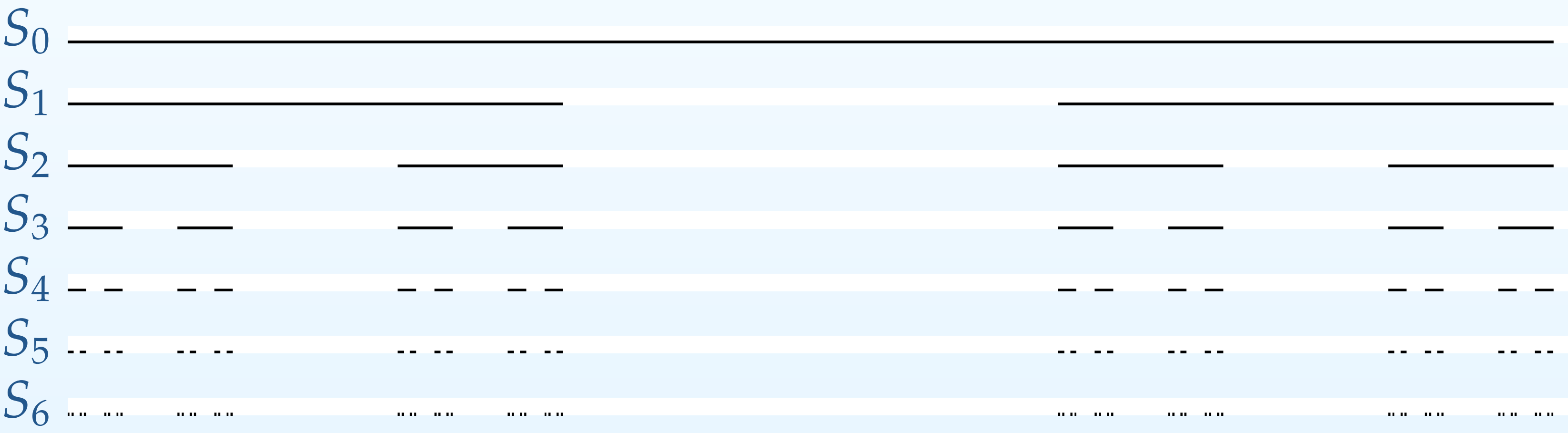
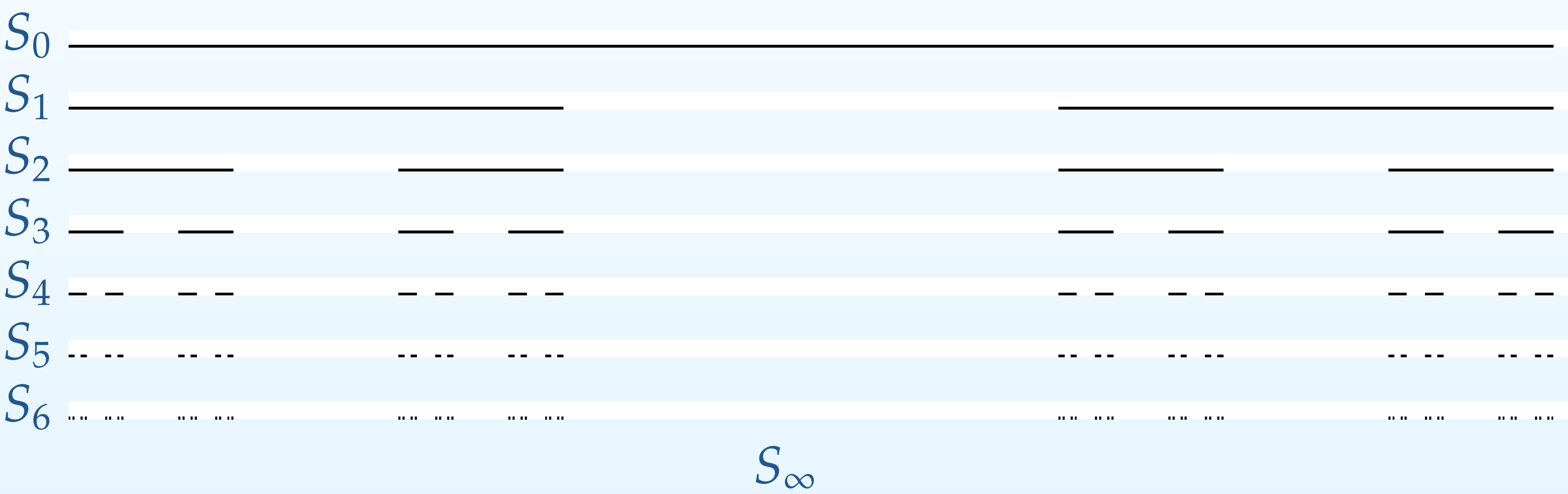


Cantor set

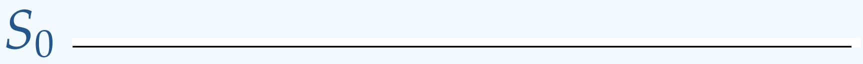
- the closed interval $S_0 = [0, 1]$
- remove its open middle third $[1/3, 2/3] \to S_1$
- remove the open middle thirds from the intervals $\to S_2$
- the limit set S_∞ is the Cantor Set



von Koch curve

von Koch curve

- A line segment S_0



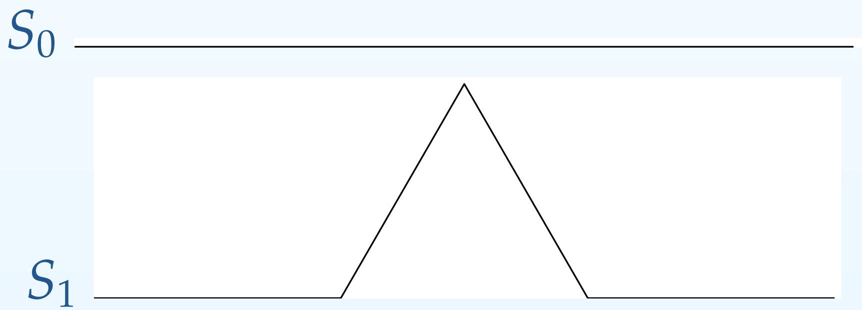
von Koch curve

- A line segment S_0 and delte the middle third of S_0

S_0

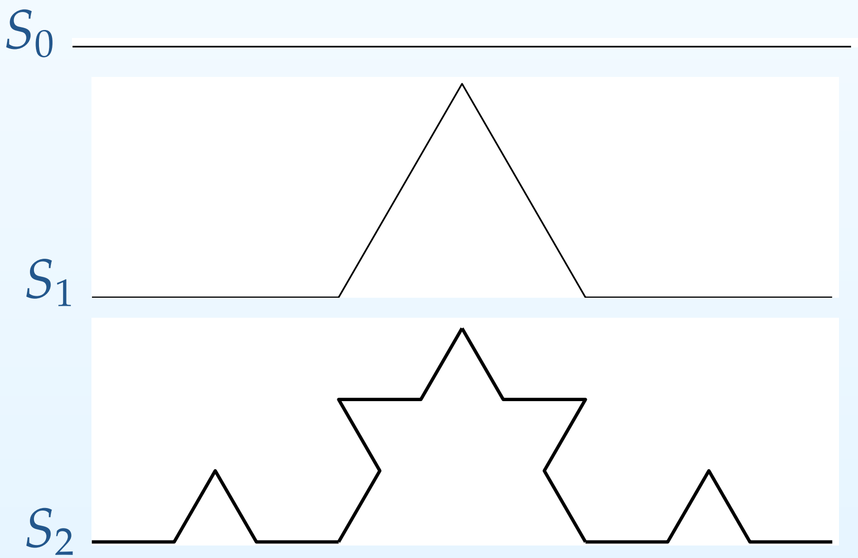
von Koch curve

- A line segment S_0 and delte the middle third of S_0
- replace it with the other two sides of an equilateral triangle $\rightarrow S_1$



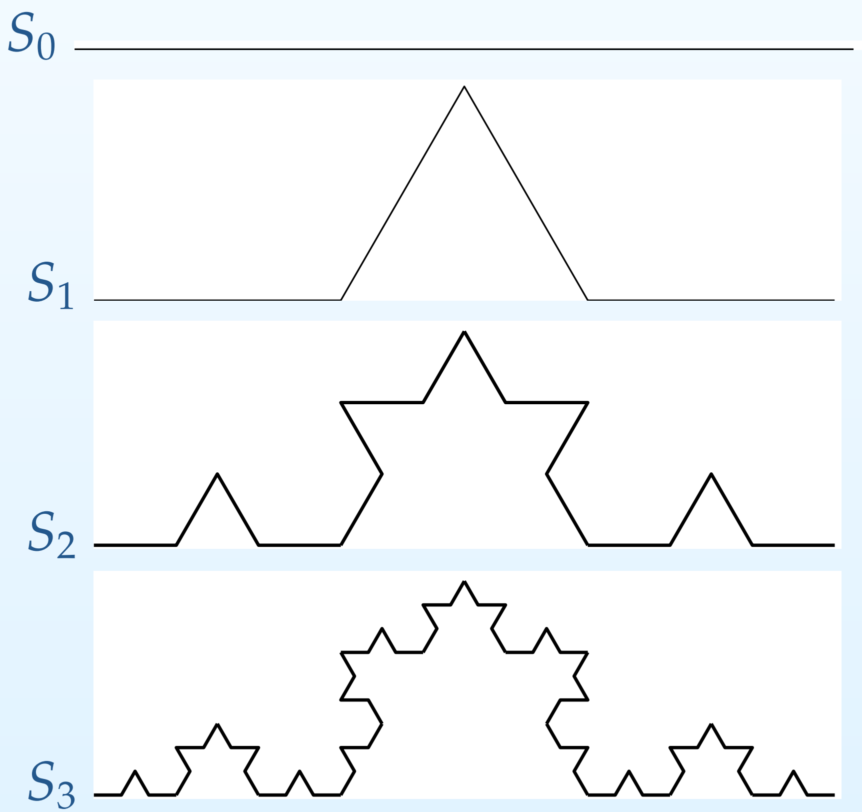
von Koch curve

- A line segment S_0 and delte the middle third of S_0
- replace it with the other two sides of an equilateral triangle $\rightarrow S_1$
- The same rule to each side $\rightarrow S_2$



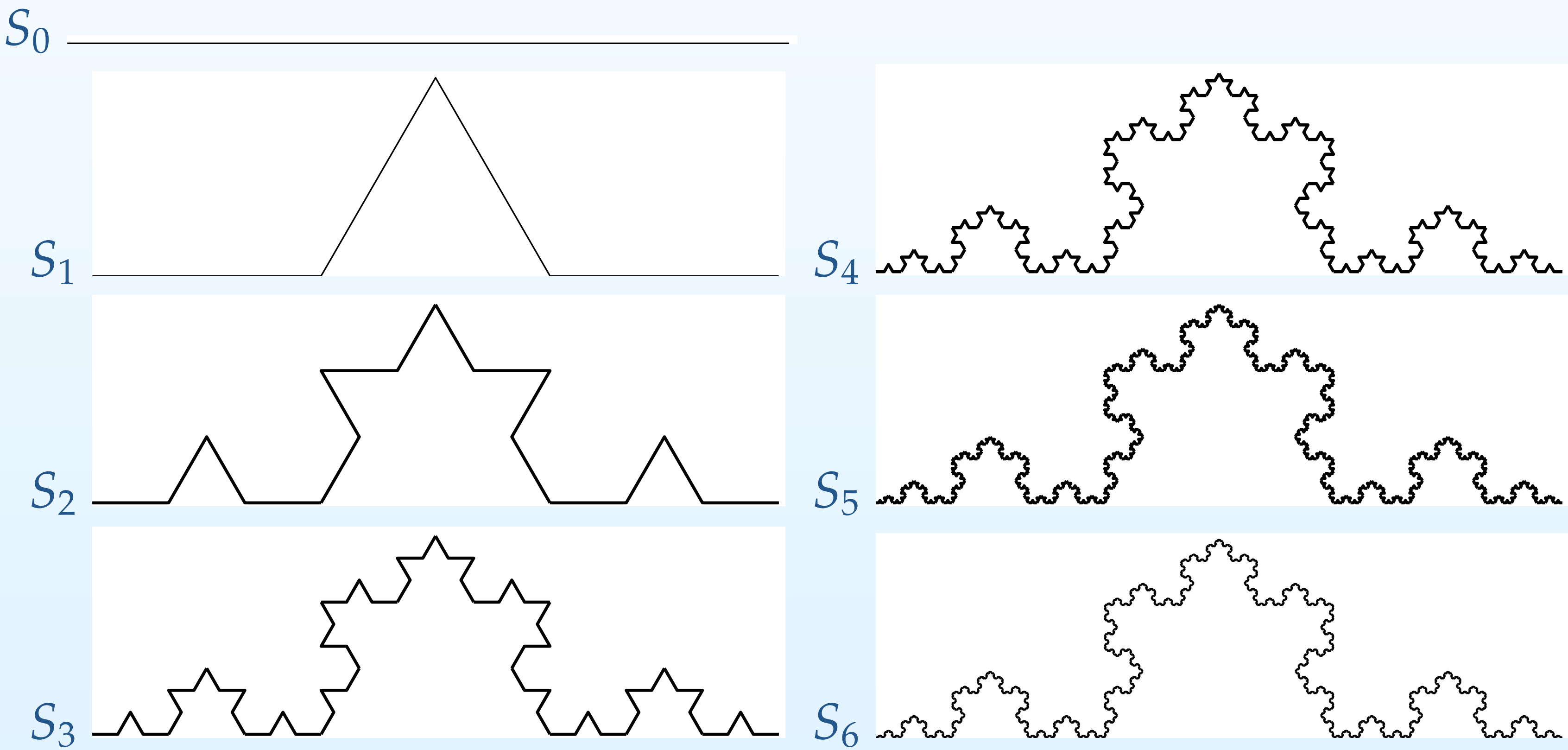
von Koch curve

- A line segment S_0 and delte the middle third of S_0
- replace it with the other two sides of an equilateral triangle $\rightarrow S_1$
- The same rule to each side $\rightarrow S_2 \rightarrow S_3 \rightarrow S_4 \rightarrow S_5 \cdots$



von Koch curve

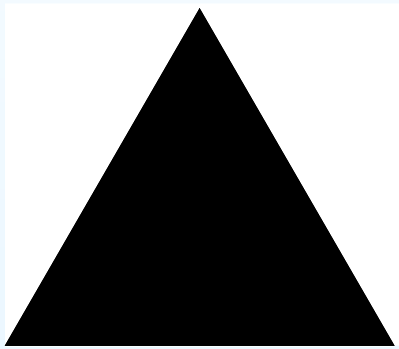
- A line segment S_0 and delte the middle third of S_0
- replace it with the other two sides of an equilateral triangle $\rightarrow S_1$
- The same rule to each side $\rightarrow S_2 \rightarrow S_3 \rightarrow S_4 \rightarrow S_5 \cdots$
- The limiting set S_∞ is von Koch curve



Sierpiński gasket

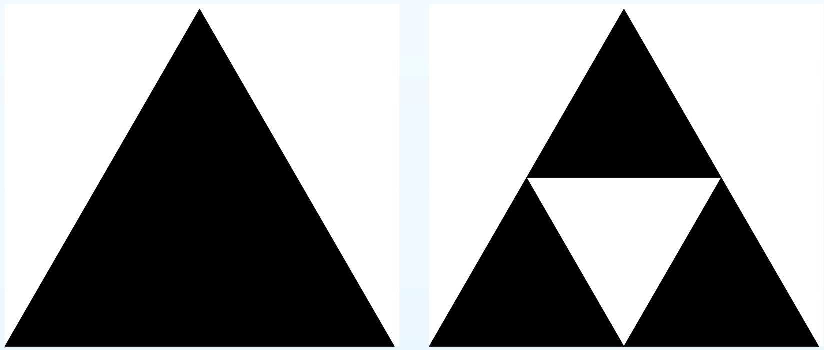
Sierpiński gasket

- An equilateral triangle



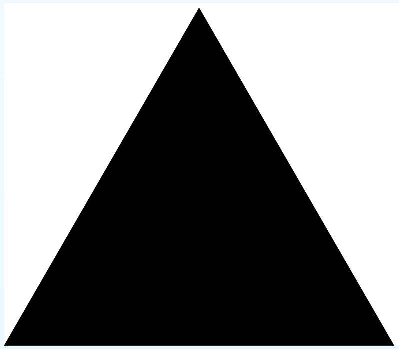
Sierpiński gasket

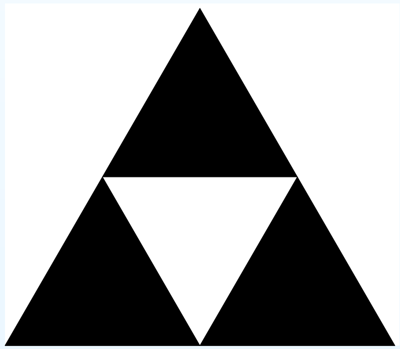
- An equilateral triangle and remove its middle (an inverted triangle)



Sierpiński gasket

- An equilateral triangle and remove its middle (an inverted triangle)
- The same rule





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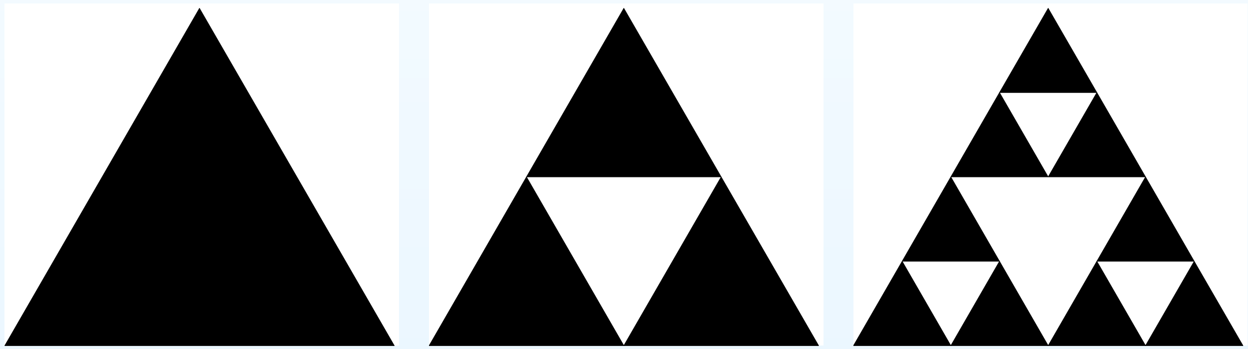

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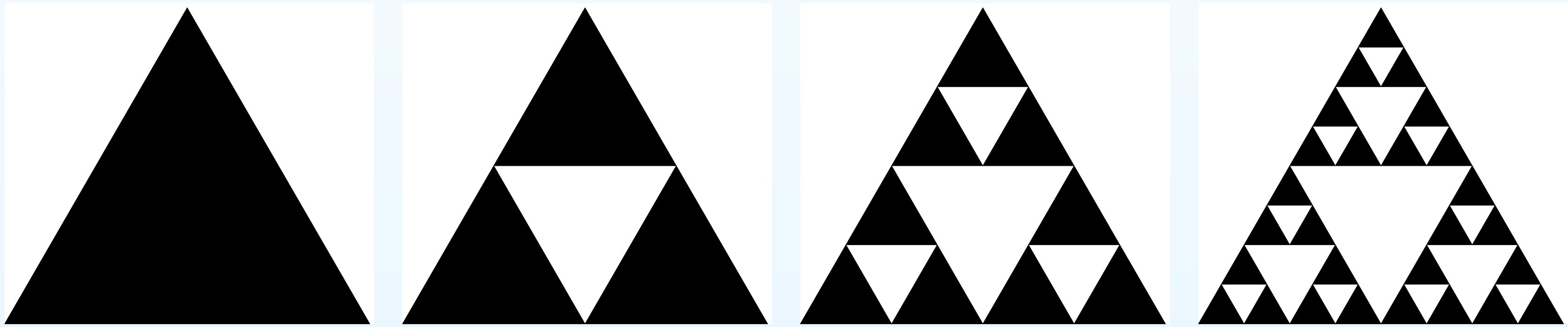
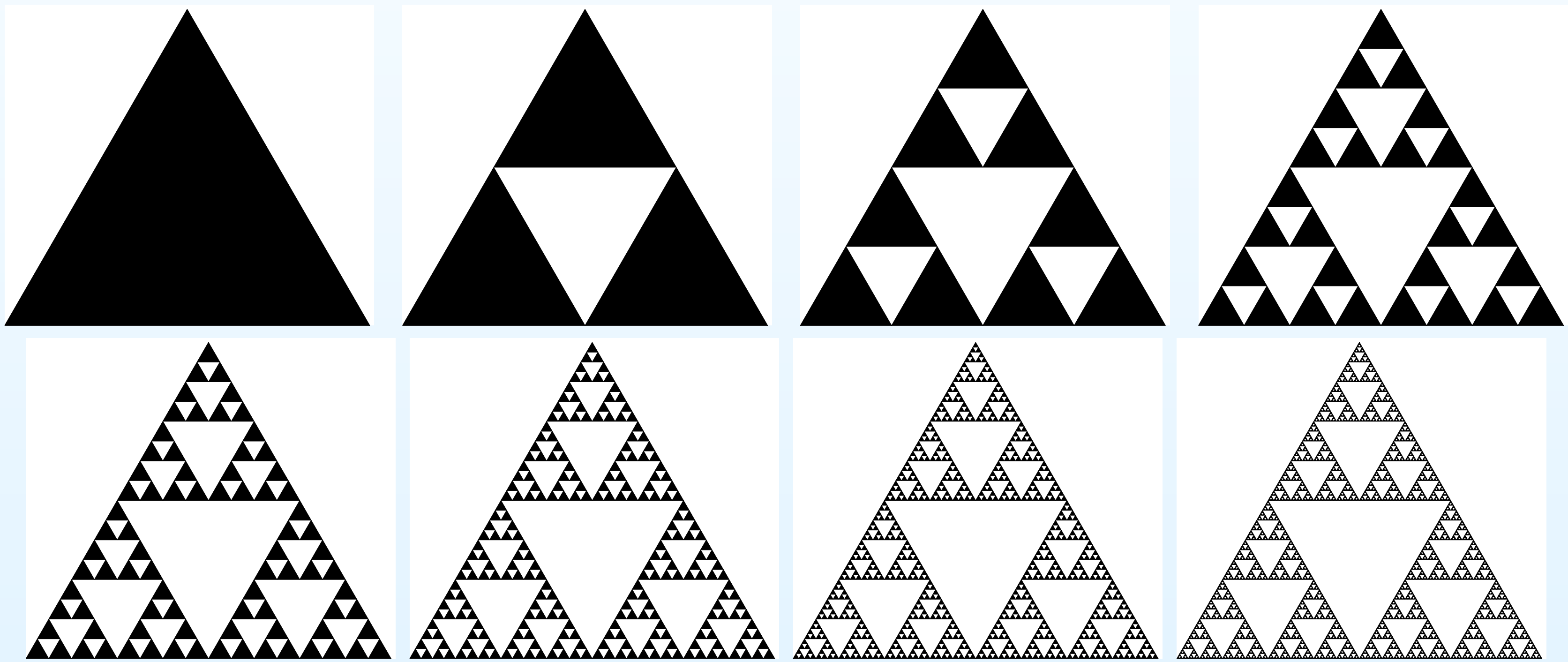


Sierpiński gasket

- An equilateral triangle and remove its middle (an inverted triangle)
- The same rule
- The limiting set is Sierpiński gasket



How to characterize fractal objects?

- Dimension
  1. Integer for lines, planes, solids...
  2. Non integer for fractal objects!

↓

What is a dimension of a set of points?
How to measure it?

What is Dimension? – Intuitively –

Line – 1 dimensional
Plane – 2 dimensional
Solid– 3 dimensional

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Definition by “Covering”

Cover a segment by a segment

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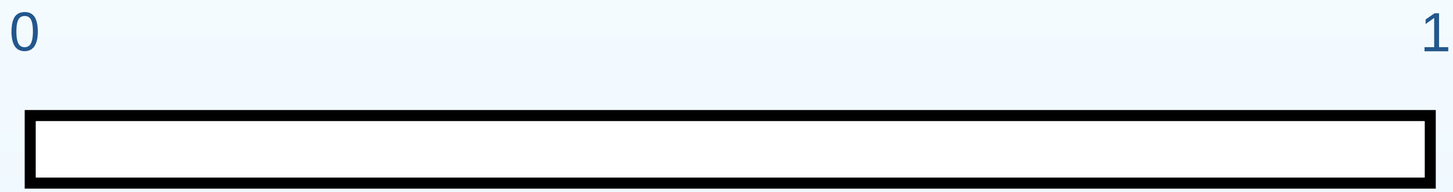

Cover a segment by a segment

Cover a segment by a segment

Cover a segment by a segment

The minimum number of small segments (scale factor $\frac{1}{2}$) for covering the unit segment = 2

Cover a square by a square

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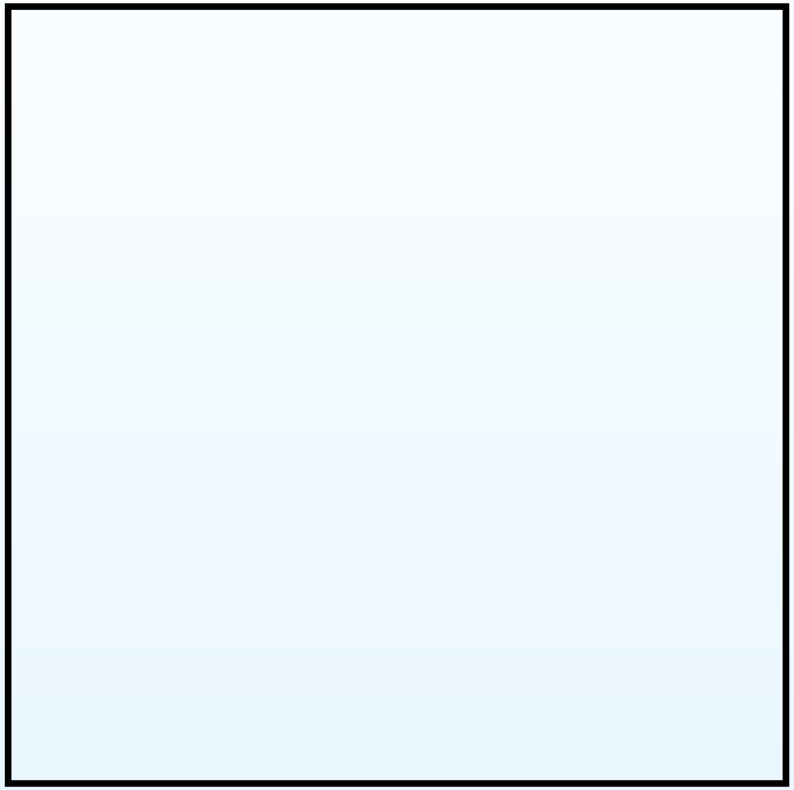

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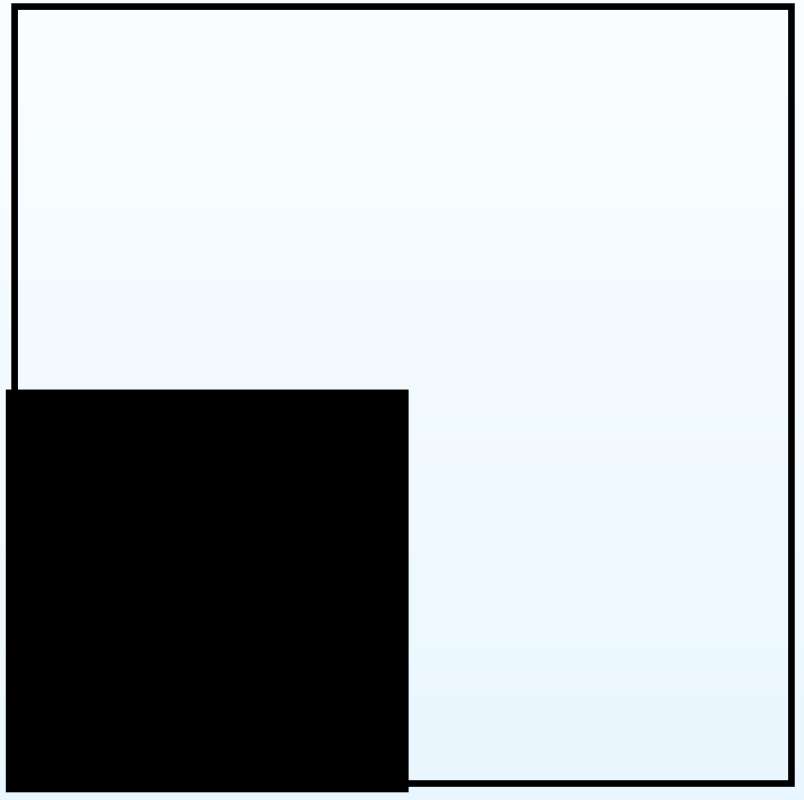

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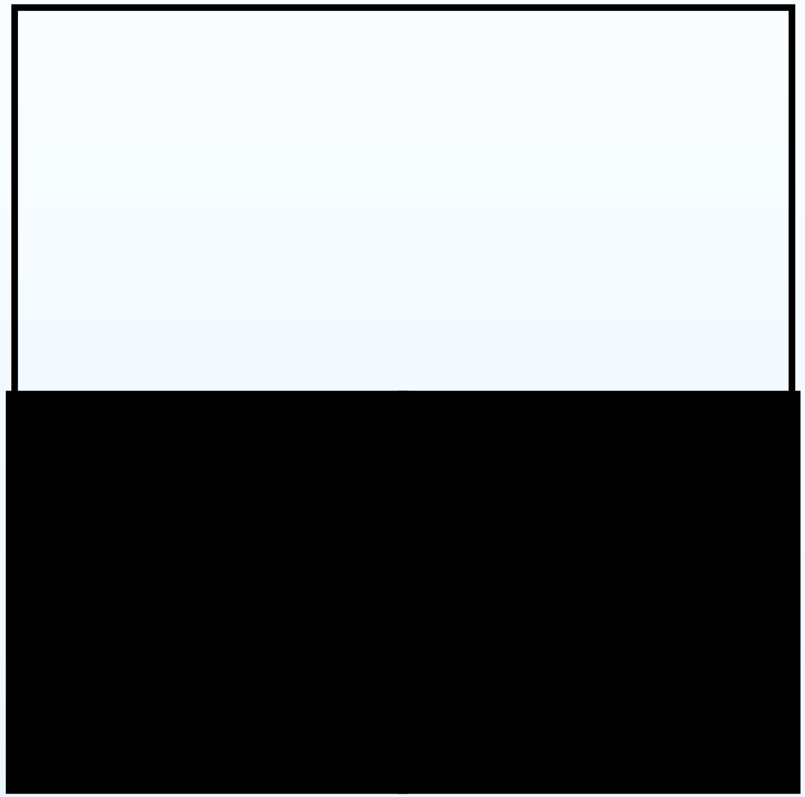

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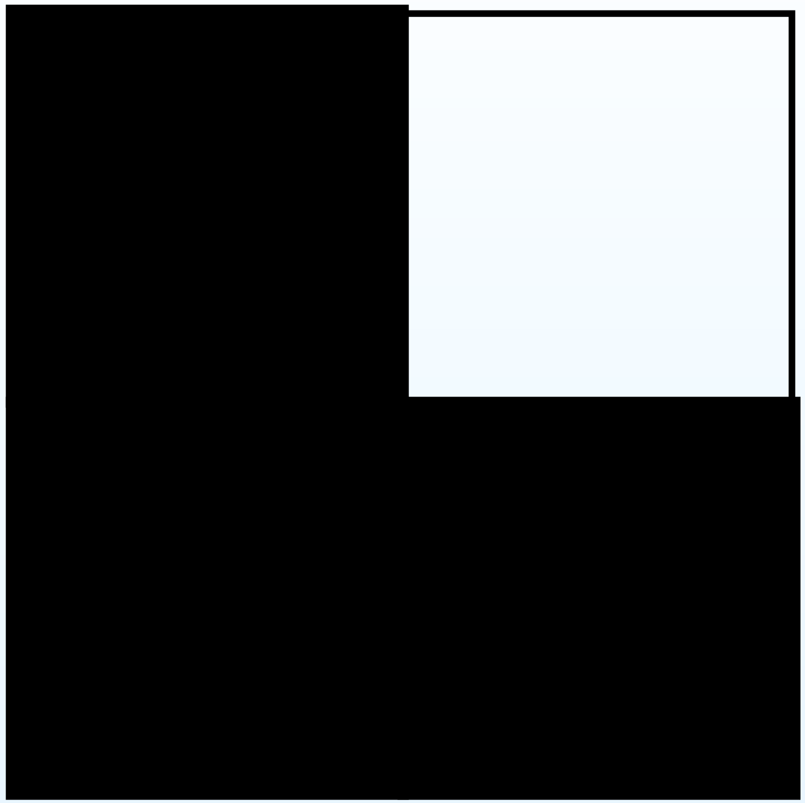

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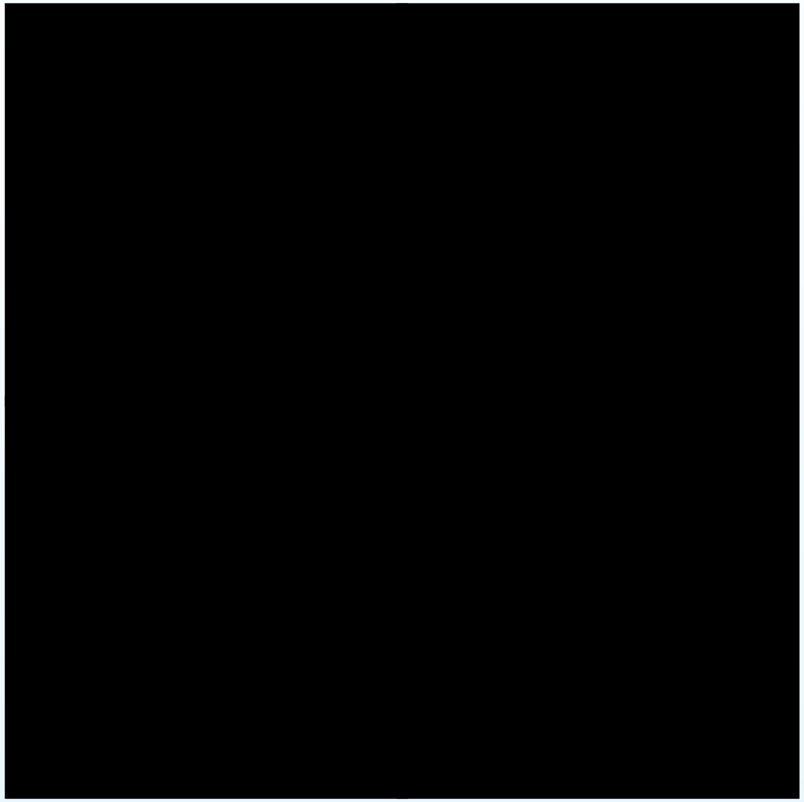

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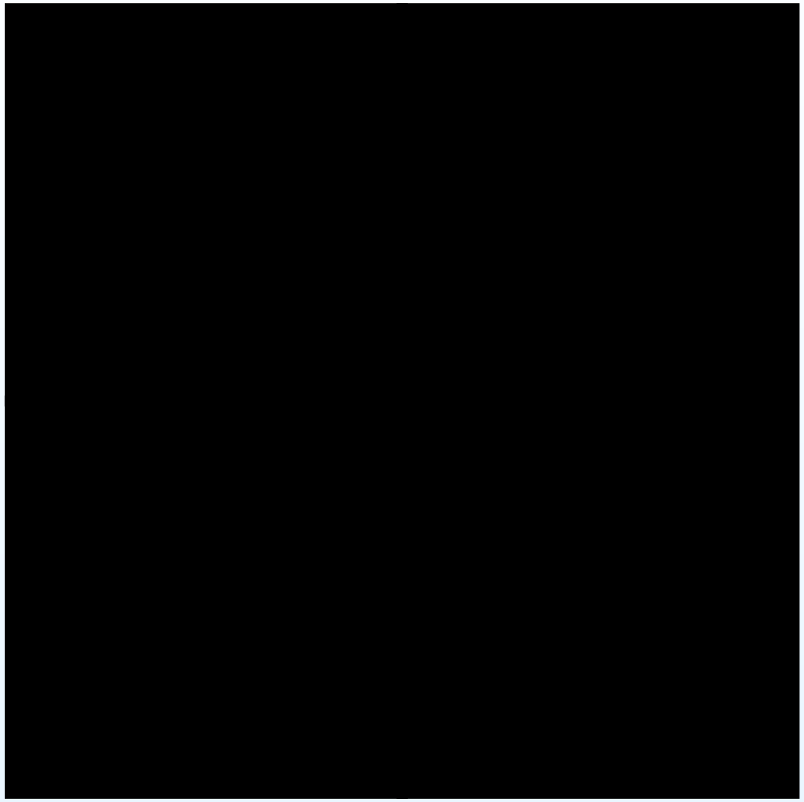
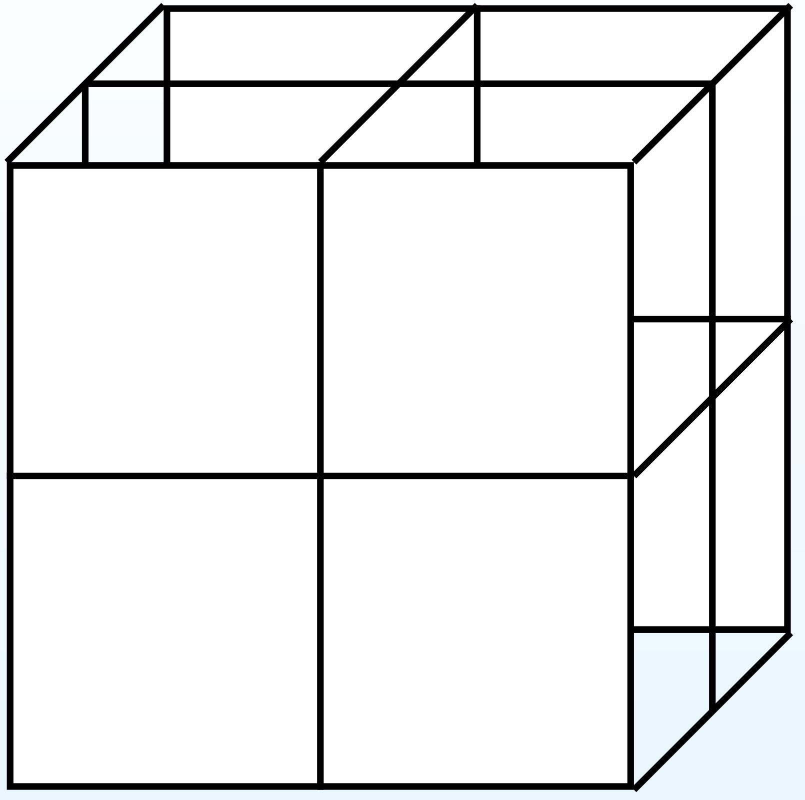


The minimun number of squares (scale $\frac{1}{2}$)

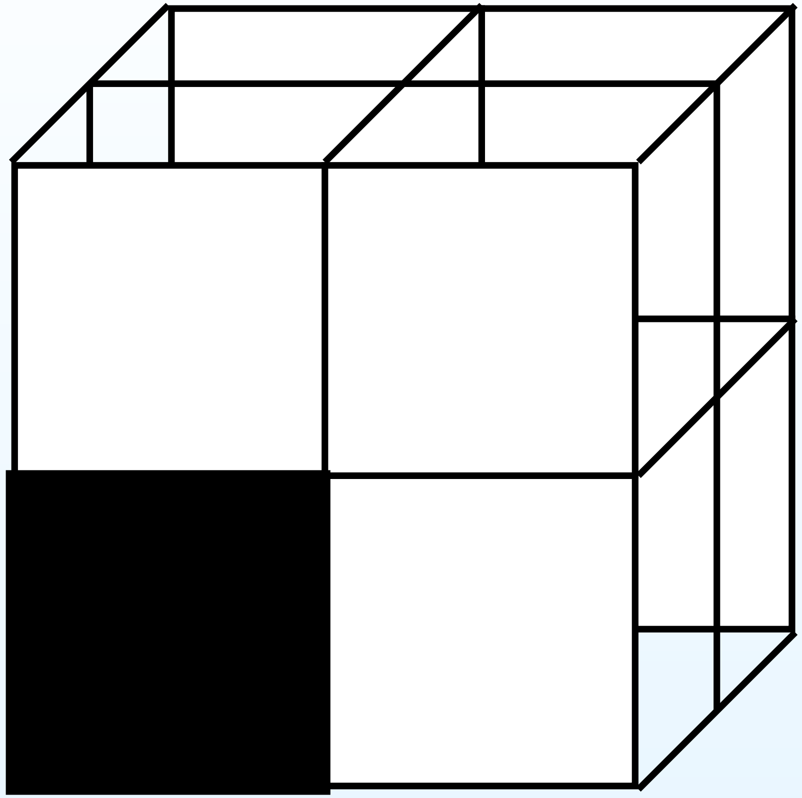
for covering the large square $= 4$

Covering a cube by a cube

Covering a cube by a cube



Covering a cube by a cube



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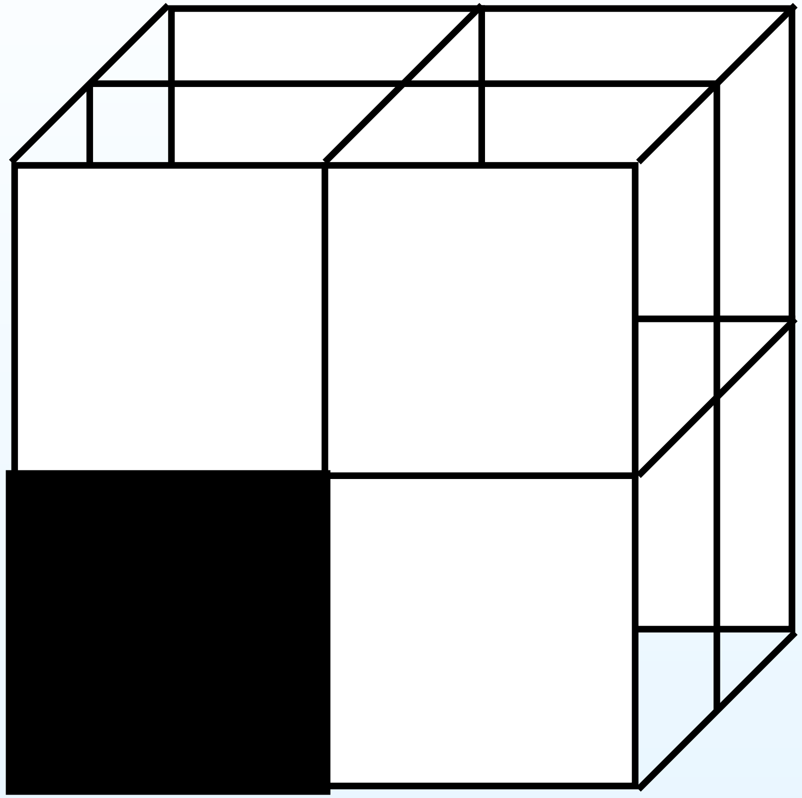

The minimum number of small cubues (scale $\frac{1}{2}$)

for covering the unit cube $= 8$

Measuring the dimension by covering

	scale	the number
Segment	$1/2$	2
Square	$1/2$	4
Cube	$1/2$	8

Measuring the dimension by covering

	scale	the number
Segment	$1/2$	$2 = 2^1$
Square	$1/2$	4
Cube	$1/2$	8

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$$N(\epsilon) = \epsilon^D$$

Definition of dimension by covering

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- Dimension D of an object is defined by the minimum number $N(\epsilon)$ of small scaled objects of $1/\epsilon$

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- So, how can we calculate the dimension of the Cantor Set?

The dimension of the Cantor set

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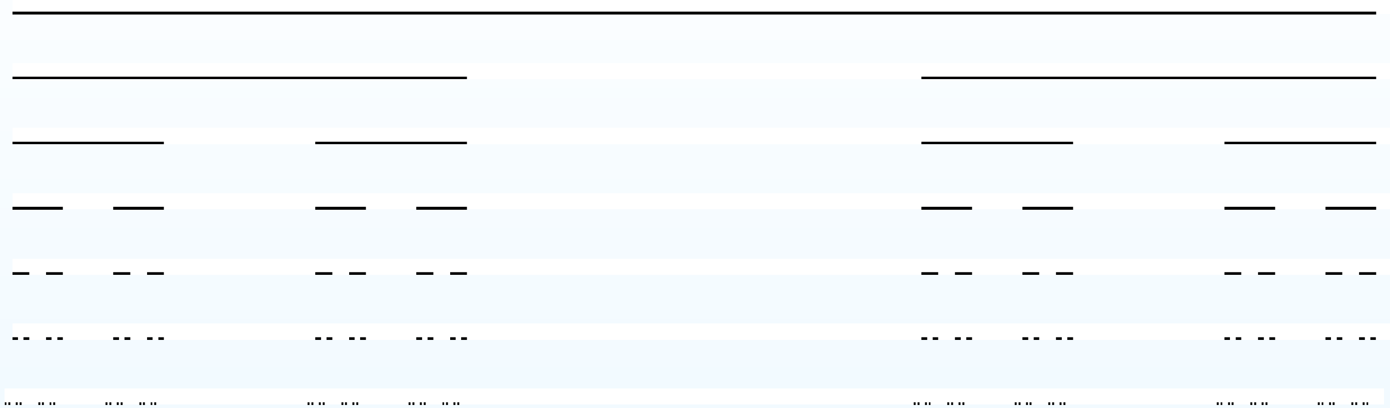
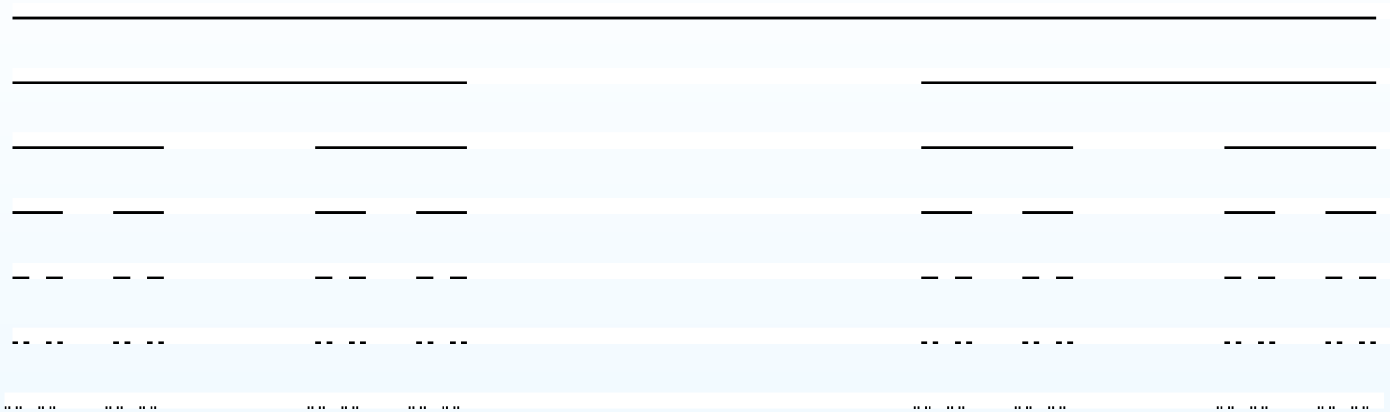
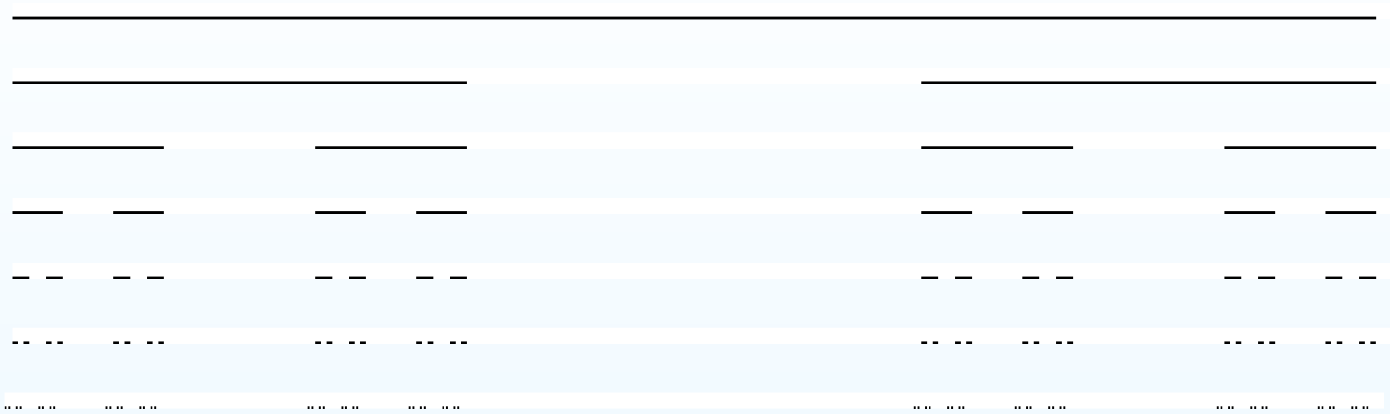


The dimension of the Cantor set



Since $\dfrac{1}{\epsilon} = \dfrac{1}{3}$ and $N(\epsilon) = 2$, we have $2 = 3^{D_C}$,

The dimension of the Cantor set



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$$D_C = \frac{\log 2}{\log 3} = 0.630929754\cdots$$

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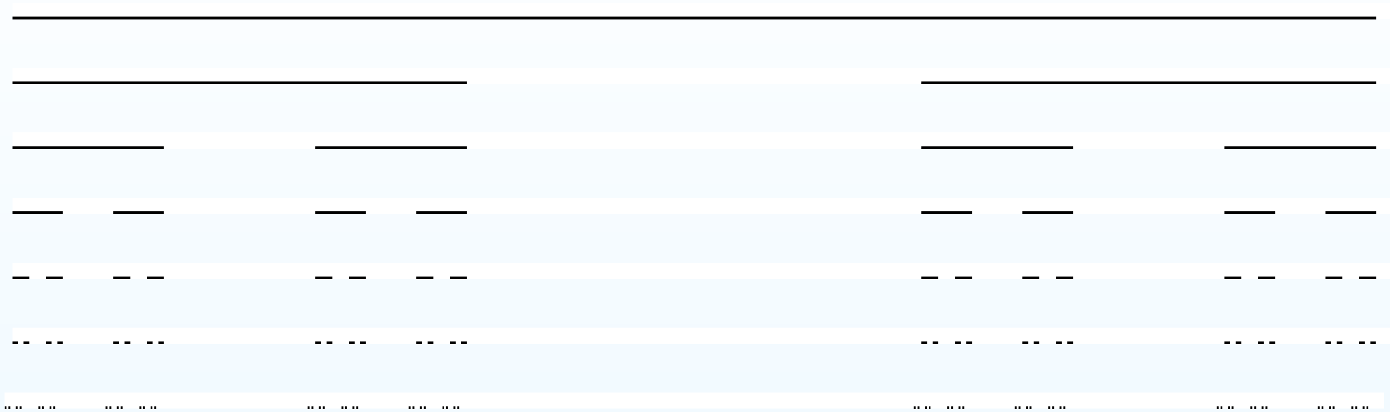

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$\longrightarrow$ Non integer!!

Noninteger dimension ...

$$0 < D_C = \frac{\log 2}{\log 3} = 0.630929754 \cdots < 1$$

- Start from a 1-dimensional object (a unit segment)
- Remove the middle thirds
- Repeat the procedure infinitely many times
- There remain many points (0-dimensional object)
- Complexity of the Cantor set is between 0 and 1

Noninteger dimension ...

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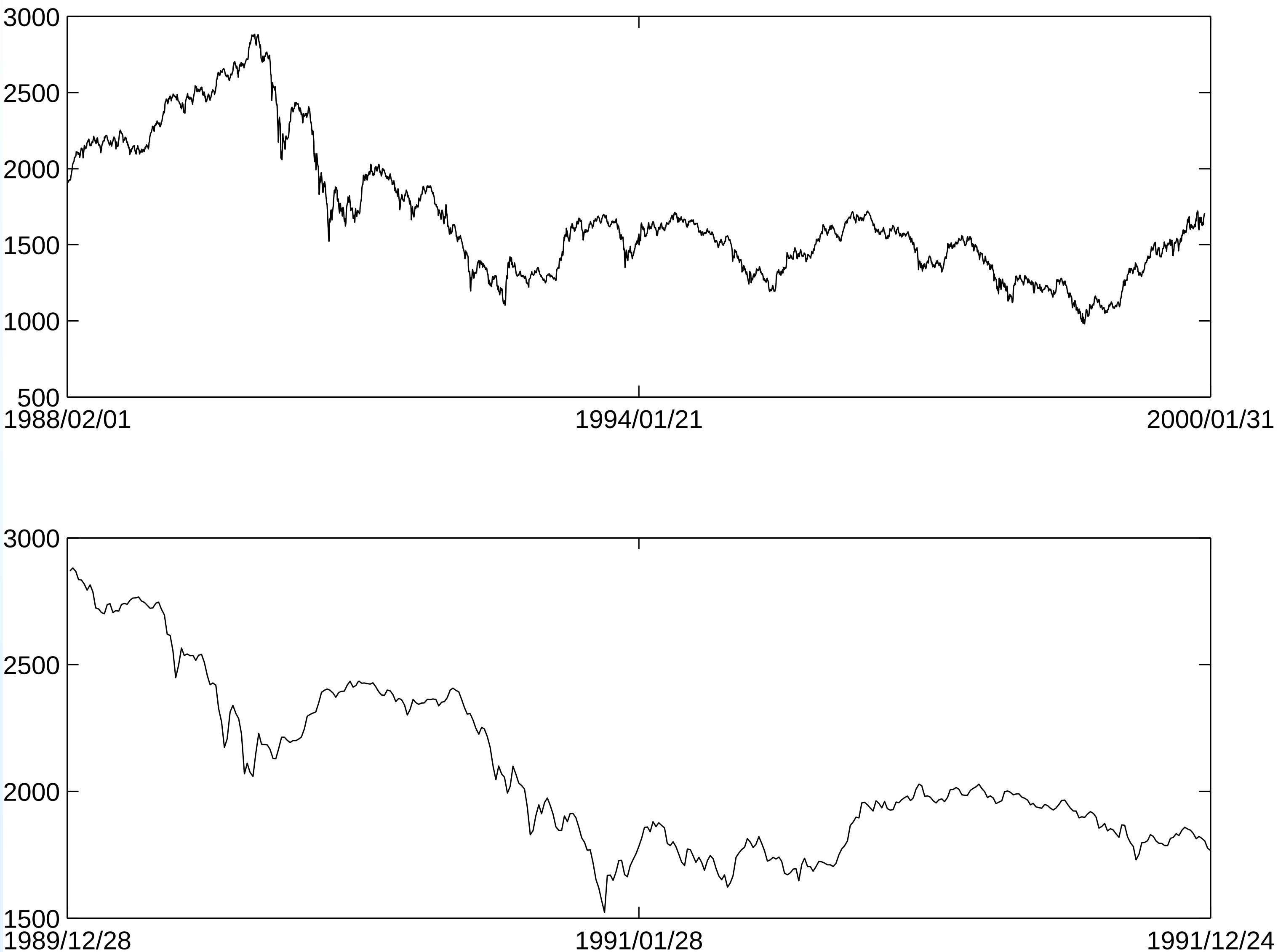
Quiz

1. Calculate the dimension of von Koch curve and Sierpiński gasket.
2. Give examples of fractals.

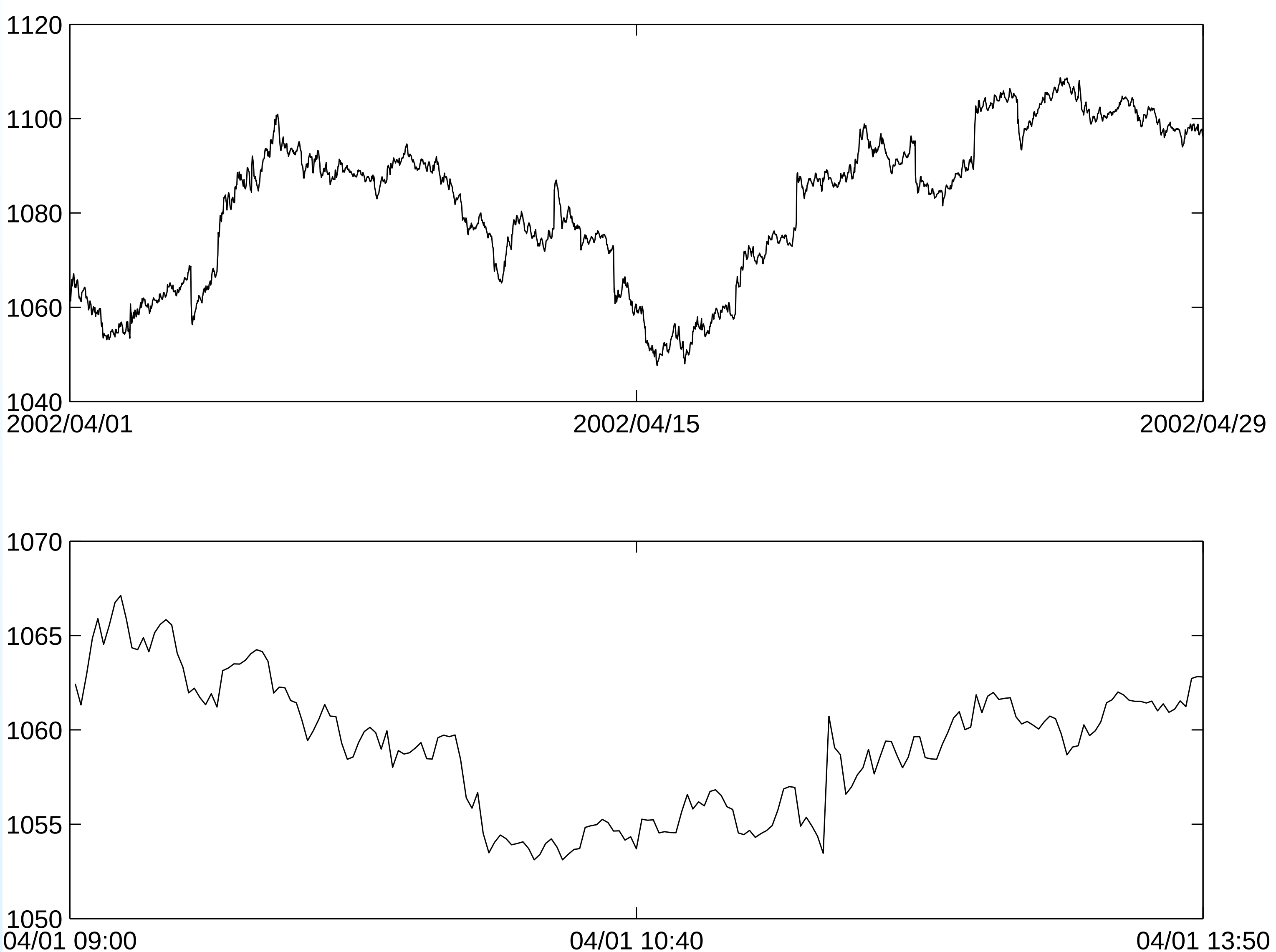
Applications of fractal

1. Analysis of financial index
   Dow Jones, Nikkei, Hang Seng, etc.
2. Packet traffic in the Internet
3. Computer graphics
4. Image processing, cinemas, games
5. Image coding (compression)

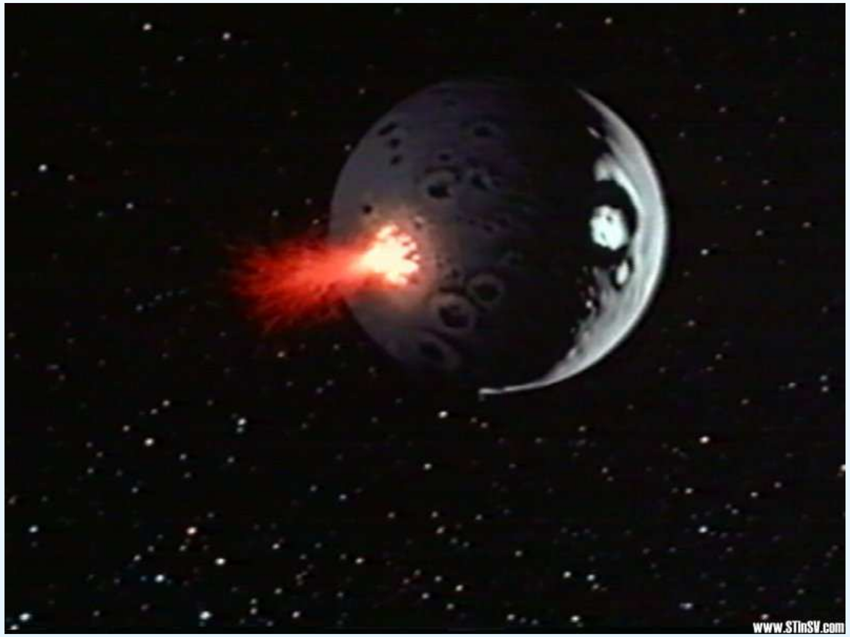
Fractal property of financial index (TOPIX)



Fractal property of financial index (TOPIX)



Star Trek II – The Wrath of Kahn – (1982)



http://www.startrek.com/

Summary of the lecture...

Clouds in the sky, thunderstorms, snow crystals,
trees, branches, leaves
coastlines, rivers, ridge lines, stones,
blood vessels, lungs, brains

↓

Fractal structure

Noninteger dimension

Home works

1. Calculate the dimension of von Koch curve and Sierpiński gasket.
2. Give examples of fractals.